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Modelling and Simulation of Ergodic Data

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Abstract

This paper presents the state space modelling and simulation of ergodic phenomena with known autocorrelation functions. The method is based on white noise filteration and for this, the state space approach to design and representation is employed. Therefore, the continuous filter characteristics and also the recurrence equations, for digital computer simulations, can be derived. The method is applied to second order Gaussian phenomena with a prescribed decaying cosine autocorrelation function. The discrete data are then simulated, on a digital computer, by exciting the filters with discrete white noise and sampling the response. The autocorrelations estimated from the simulated phenomena are plotted and compared with the truely

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expected values.

Introduction

The simulations of ergodic phenomena with a prescribed autocorrelation function may be required for a variety of purposes. However, the desired autocorrelation function should be capable of representing random data. It is known that for zero mean data, the autocorrelation decays as the time delay increases. [1]

The generation of linear processes is based on passing white noise through linear filters. The filter characteristics should be derived according to the desired autocorrelation function of the output signal. The differential equation modelling of time series is well know and has been discussed in Faghih. [2],[3]

After obtaining the filter transfer function and before attempting to simulate on a digital computer, one has usually to struggle for the derivation of the recurrence equations. The state space approach presented in this paper, leads to matrix recurrence equations, suitable for digital computer simulations.

In face , filters are designed in the continuous form, which when excited with continuous white noise should output the desired continuous data. When the filters are excited with discrete white noise, however, the discrete signals with any time spacings may be obtained by sampling the response. The martix recurrence equations for digital computer simulations are, hence, derived.

The method is applied to generate a second order Gaussian process with a decaying cosine autocorrelation function. Samples of realizations are shown. The estimates of the autocorrelation functions, obtained from the simulated data, are also shown and compared with the expected functions.

Generation of White Ergodic Data

Theoretically , white noise is a random process whose autocorrelation is zero everywhere, except at zero time delay; its power spectrum is also constant for all frequency range. The second property justifies the term ' white noise' by analogy with the optical spectrum of white light.

Most of the computer languages and even some desk calculators include algorithms or functions for generation of random sequences, in the range (0,1) , with a uniform probability distribution. Today, the most successful generation of these sequences is according to the multiplicative congruential method [4]. In this metod, the two following multiplicative sequences are generated:

$$x_{r+1} = ax_r \qquad (\text{Mod - M}) \qquad (1)$$

$$y_{r+1} = by_r \qquad (\text{Mod - M}) \qquad (2)$$

where Mod-M stands for Modulo-M, implying that x_{r+1} and y_{r+1} are the remainders of the integer division of ax_r and by_r , respectively, by M. A sequence of random numbers, U_{r+1} is then formed in the range (0,1) , using:

$$U_{r+1} = \text{Fraction } [(x_{r+1} + Y_{r+1})/M] \quad (3)$$

a,b and M are constants and fraction (...) denotes that the quantities inside the brackets are divided by M and the integer part is taken away; i.e. the fraction part is only used.[4]

The following constants, used for a, b and M, can give a very long sequence of 10^{12} random numbers, before the same sequence is repeated: [5]

$$a = 3^{15} \;,\; b = 5^9 \;,\; M = 2^{46}$$

The seed number for the start of the sequence, (x_0, y_0), can also be chosen as a seven digit number, containing numbers 1 to 7 in any order. [5]

White noise generation, with desired probability density function, is usually approached using the uniform random sequence, U_r, as a basis. For example, to obtain a Gaussian random sequence, the method of Box and Muller [6] may be used:

$$w_{r-1} = (-2 \ln U_{r-1})^{\frac{1}{2}} \cos (2\pi U_r) \quad (4)$$

$$w_r = (-2 \ln U_{r-1})^{\frac{1}{2}} \sin (2\pi U_r) \quad (5)$$

The sequence W_r forms a standard Gaussian (zero mean and unit variance) white noise [5], [7]. If Gaussian white noise Z, with arbitrary mean (μ) and variance (σ^2) is required, then the following transformaion can be applied:

$$Z = \sigma w_r + \mu \tag{6}$$

Modelling and Simulation of Ergodic Data

Ergodic data with certain prescribed autocorrelation functions (or power spectra) may be simulated by passing white noise through linear filters. A linear filter (or system) is the physical realization of a linear differential (or in discredte case, difference) equation. The simulated process is, hence, effectively represented by a differential or difference equation, being obtainable for the known linear filter characteristics. [7]

Consider a linear filter with the Laplace transfer function G(s), or frequence response $G(j\,w)$, where the Laplace operator s is replaced by $j\,w$; w being the circular frequency and j is the complex number $\sqrt{-1}$. It can be shown [7] that the following relationship holds between the input and output spectral densities:

$$S_y(w) = G\,(j\,w)\,G\,(-j\,w)\,S_u\,(w) \tag{7}$$

where $S_u\,(w)$ and $S_y(w)$ are the input and output signal spectral densities, repectively. $G\,(-jw)$ is the complex conjugate of $(j\,w)$ and, hence, $G(j\,w)G(-j\,w)$ is $|G(j\,w)|^2$ which is a real quantity.

The input signal can be chosen as a zero mean, unit variance white noise, for which:

$$S_u(w) = 1 \tag{8}$$

Equation (7) will, therefore, give:

$$S_y(w) = G\,(j\,w)\,G\,(-j\,w) = |G(j\,w)|^2 \tag{9}$$

Equation (9) suggests an approach, as well as limitation, to generating ergodic data with a prescribed autocorrelation function (or spectal density). If the ergodic data is required to have an autocorrelation function R(τ), its (mathematical, two sided) spectral density $S_y(w)$ can be found by Fourier transformation of R(τ), [1]; i.e.:

$$S_y(w) = \int_{-\infty}^{\infty} R(\tau) \exp(-j\, w\, \tau)\, dt \qquad (10)$$

This spectral density has to be factorized into complex conjugate terms to obtain the required filter frequency response G(j w), or equivalently a function G(j w) has to be found such that $|G(j\,w)|^2$ equals $S_y(w)$. It is shown [8] that a linear filter design is possible only if $S_y(w)$ is a rational function in terms of w^2; i.e. if $S_y(w)$ is expressible as a quotient of two polynomials including even powers of w only. If a power spectrum, which may have been obtained experimentally for example, does not meet this requirement, it has to be approximated by a rational function of w^2. Laning and Battin [8] have discussed methods of obtaining a least-square approximate in such situatioins. The examination of non-linear filter design shows that similar limitations would apply to this [9]. For a non - linearity , however, an additional problem is also imposed. That is, while linear systems preserve the probability density of the input signal, non linear systems do not. It seems that, the design of a filter for an arbitrary prescribed power spectrum is not yet possible. [7].

However, when the required linear filter frequency response, $S_y(w)$, is obtainable, jw may be replaced by the Laplace operator s to give the filter transfer function, G(s), [10].

Then from the transfer function, G(s), the recurrence equations for simulation on a digital computer can be obtained. The state space approach to this will be presented in the next section.

State Space Approach

The Laplace transfer function of the required linear filter is generally expressible as a quotient of two polynomials in terms of powers of the Laplace operator s, [8]; i. e., as:

$$G(s) = \frac{c_m s^m + c_{m-1} s^{m-1} + \dots + c_o}{s^n + a_{n-1} s^{n-1} + \dots + a_o} \qquad (11)$$

The system represented by the above transfer function, can be modelled in the state space form, by the following matrix differential equations [10],

$$\chi = A\underline{x} + \underline{b}u \qquad (12)$$

$$y = c\underline{x} \qquad (13)$$

where x is the state vector, u is the input to the system, y is the system output and the system matrix A is in the companion form, i.e.,

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \qquad (14)$$

$$b = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \qquad (15)$$

$$c = [c_0 \quad c_1 \quad c_2 \quad \ldots \quad c_m \quad 0 \quad 0 \ldots \quad 0] \qquad (16)$$

For a system of order n, as in equation (11), A is also a square matrix of order n and vectors b and c each contain n elements.

It can be shown [11], that the digital computer solution to equations (12) and (13) would appear as following recurrence equations:

$$\underline{x}(kT) = G(T)\underline{x}\,[(k-1)\,T] + H\,(T)u\,[(k-1)\,T] \qquad (17)$$

$$y(kT) = c\underline{x}\,(kT) \qquad (18)$$

where T is the sampling period, k is an integer and matrices G(T) and H(T) follow:

$$G(T) = e^{AT} \tag{19}$$

$$H(T) = [\int_0^T e^{AT}\, dt]\, \underline{b} \tag{20}$$

Making use of the following eigen - transformations [10],

$$e^{AT} = We^{\Lambda t}\, W^{-1} \tag{21}$$

equations (19) and (20) will lead to:

$$G(T) = W\, \phi\, W^{-1} \tag{22}$$

$$H(T) = W\Lambda^{-1}\, (\phi - I)\, W^{-1}\, \underline{b} \tag{23}$$

where:

$$\phi = e^{AT} \tag{24}$$

$$\Lambda = \text{diag}\, \{\lambda_1, \lambda_2, \ldots\ldots, \lambda_n\} \tag{25}$$

for distinct eigenvalues $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ of system matrix A. For the case of mutiple eigenvalues, equivalently, Jordan canonical form can instead be used. The eigenvector matrix W is the matrix with columns as the corrosponding eigenvectors, i.e.

$$W = [\underline{w}_1 : \underline{w}_2 : \ldots\, \underline{w}_n] \tag{26}$$

Where w_i are eigenvectors of the system matrix A. Matrix I is also the unity matrix.

Substituting from equations (22) and (23), would yield equations (17) and (18) as:

$$\underline{x}(k) = W\, \phi\, W^{-1}\, \underline{x}\, (k-1) + W\Lambda^{-1}\, (\phi - I)\, W^{-1}\, \underline{b}u(k-1) \tag{27}$$

$$y(k) = c\underline{x}(k)$$

Where x(k) and y(k) are the same as x(kT) and y(kT), correspondingly, implying sample values at times KT. It is interesting to note that [10],

$$\Lambda^{-1} = \text{diag}\ \{\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}\} \quad (29)$$

$$\phi = e^{\Lambda T} = \text{diag}\ [e^{\lambda_1 T}, e^{\lambda_2 T}, \ldots, e^{\lambda_n T}\} \quad (30)$$

which makes the digital computer simulation of equations (27) and (28) an easy task. Using discrete white noise as the input u(k), the desired random precess y(k) can be generated by equations (17) and (28). This will be further illustrated by introducting a numerical example in the next section.

Numerical Example:

Second Order Gaussian Phenomena

As a numerical example, consider a second order Gaussian proces with the following autocorrelation function:

$$R(\tau) = e^{-\tau} \cos \pi \tau \quad (31)$$

Substituting equation (31) in equation (10) and integraing, will yield:

$$S_y(w) = \frac{2[(1+\pi^2) + w^2]}{[1+(\pi + w)^2]\ [1 + (\pi - w)^2]}$$

This can be factorized into complex conjugate terms as:

$$S_y(w) = \frac{\sqrt{2}\,(\sqrt{1+\pi^2} + jw)}{[1+\pi^2 + (jw)^2] + 2(jw)} \cdot \frac{\sqrt{2}\,(\sqrt{1+\pi^2} - jw)}{[1+\pi^2 + (-jw)^2] + 2(-jw)} \quad (33)$$

Comparison of equations (33) and (9) gives:

$$G(jw) = \frac{\sqrt{2}\,(\sqrt{1+\pi^2} + jw)}{[1+\pi^2 + (jw)^2] + 2(jw)} \quad (34)$$

Replacing jw by the Laplace operator s, will render the required filter transfer function as:

$$G(s) = \frac{\sqrt{2}\,(s + \sqrt{1 + \pi^2})}{s^2 + 2s + (1+\pi^2)} \quad (35)$$

It is observed that the above transfer function represents a second order system. Comparing equations (35) and (11), shows that the above system may be represented in the state space form according to equations (12) and (13), where,

$$A \begin{bmatrix} 0 & 1 \\ -1-\pi^2 & -2 \end{bmatrix} \quad (36)$$

$$b= \begin{bmatrix} 0 \\ 1 \end{bmatrix} , \quad c= [\sqrt{2(1+\pi^2)} \quad \sqrt{2}] \quad (37)$$

For matrix A in equation (30), the eigenvalues are computed as:

$$\lambda_{1,2} = -1 \pm j\pi \quad (38)$$

which can be substituted in equations (29), (30) and then in equation (27). It is then proceeded to compute the corresponding eigenvectors in order to determine the eigenvector martrix W and its inverse W^{-1} , for substitution into equation (27). The discretc values of Gaussian white noise can also be obtained from euqationa (4) and (5) and then used for u(k) in euqation (27).

Finally the discrete values of the required second order Gaussian process, y(k) , may be computed from equation (28), by using the values in equation (37) and x(k) as obtained from equation (27). The discretc values of Gaussian white noise can also be obtained from euqations (4) and (5) and then used for u(k) in euqation (27).

In order to start the computation, an arbitrary vector may be assigned to x (0).

It should be noted that the discrete process, y(k), is sampling of the corresponding continuous process at time intervals of T, as chosen and used in equation (30). As numerical examples, the process was simulated, on a digital computer, with T=0.05 and T= 0.1 seconds. A realization of the process is shown for each case in Figures 1 and 2. Further, samples of 30000 values of the discrete process was simulated, to estimate the autocorrelation functions. For both cases, the estimated autocorrelation coefficients are shown and compared with the true function in

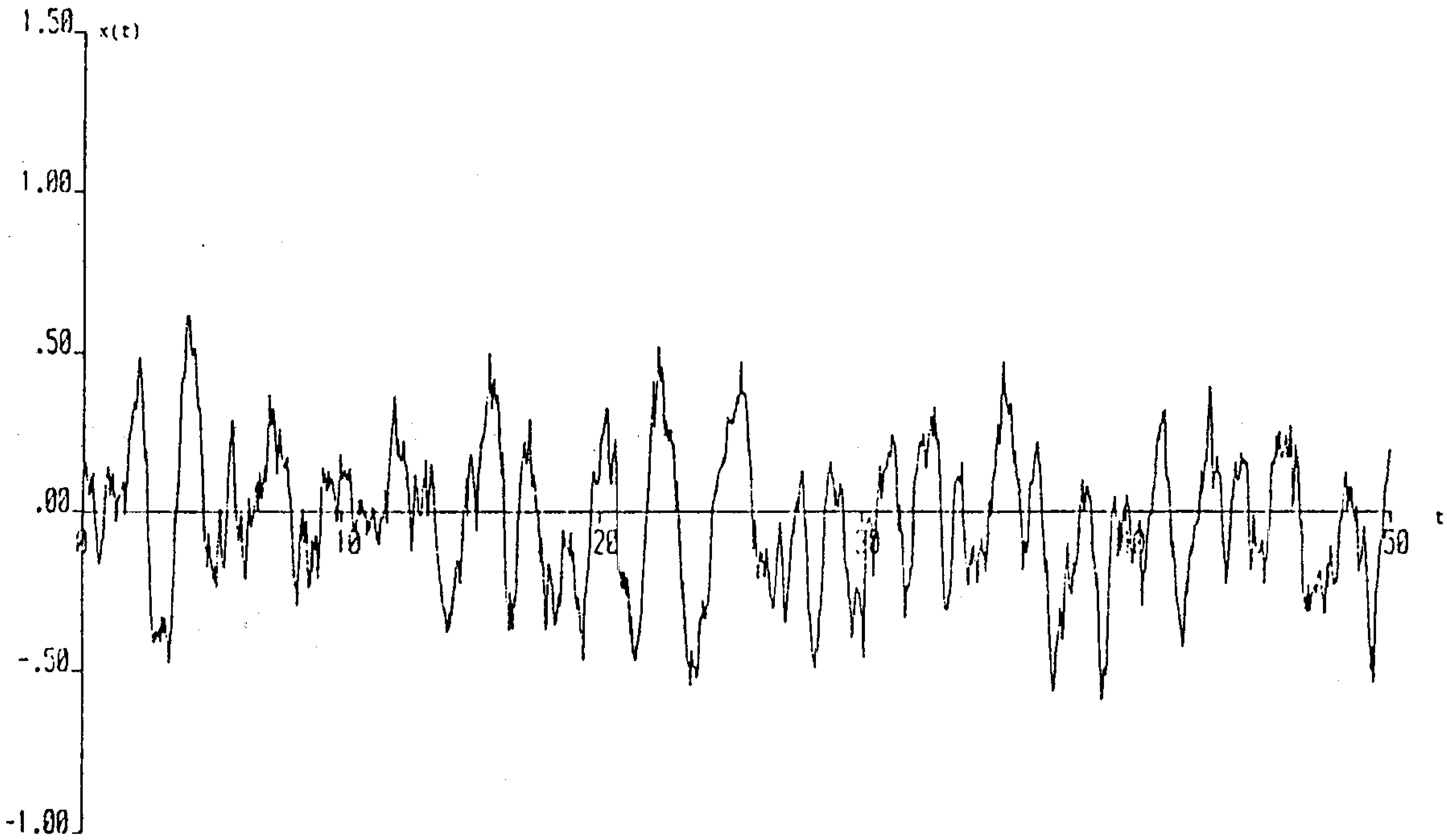

Figure 1- The random data with autocorrelation function:
$R(r) = \exp(-\tau)\cos(\pi\tau)$,
simulated with a time interval $\Delta\tau = 0.05$

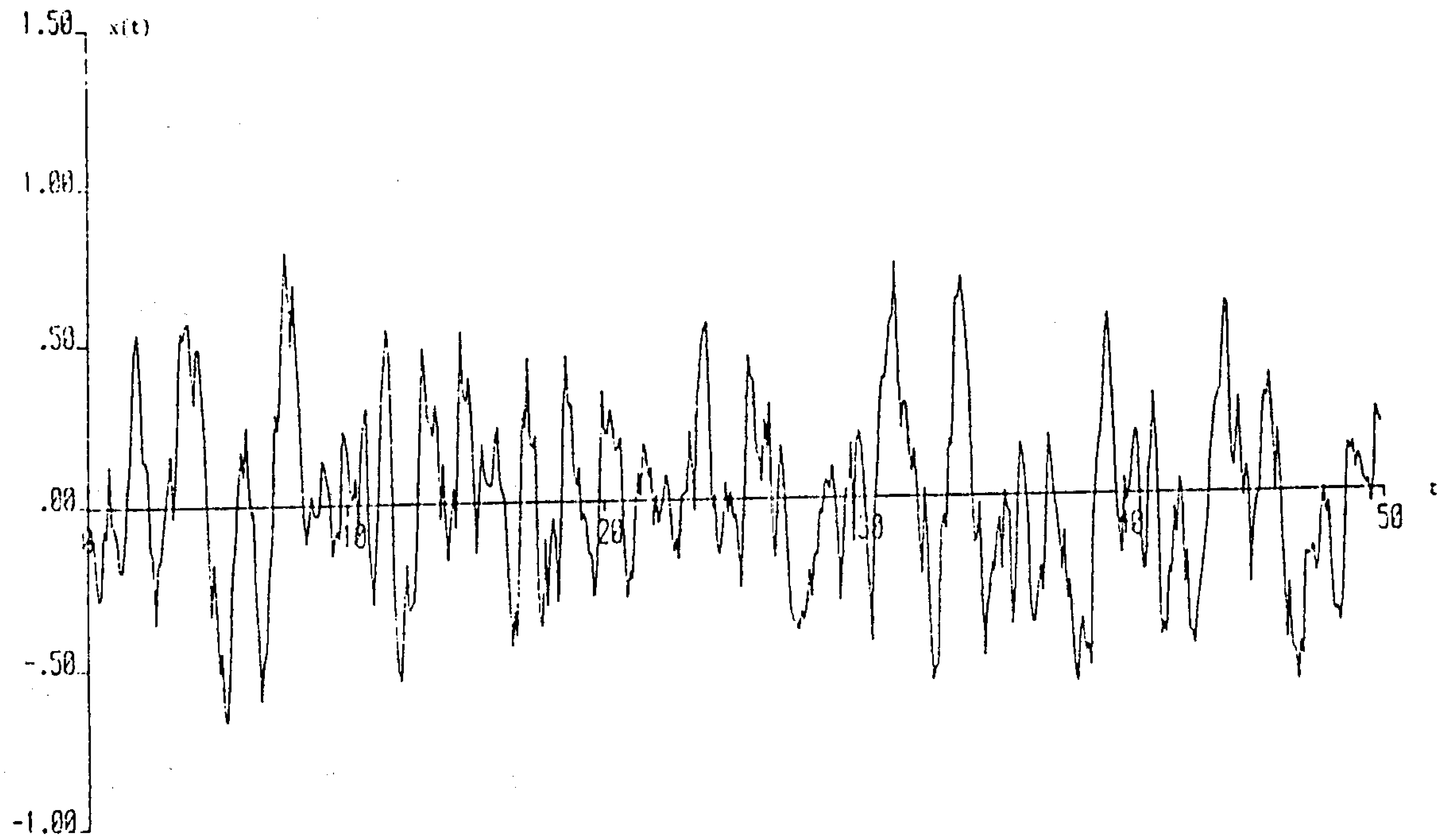

Figure 2- The random data with autocorrelation function:
$R(r) = \exp(-\tau)\cos(\pi\tau)$,
simulated with a time interval $\Delta\tau = 0.1$

Figures 3 and 4, which exhibit good agreement of the estimated coefficients with the true curves.

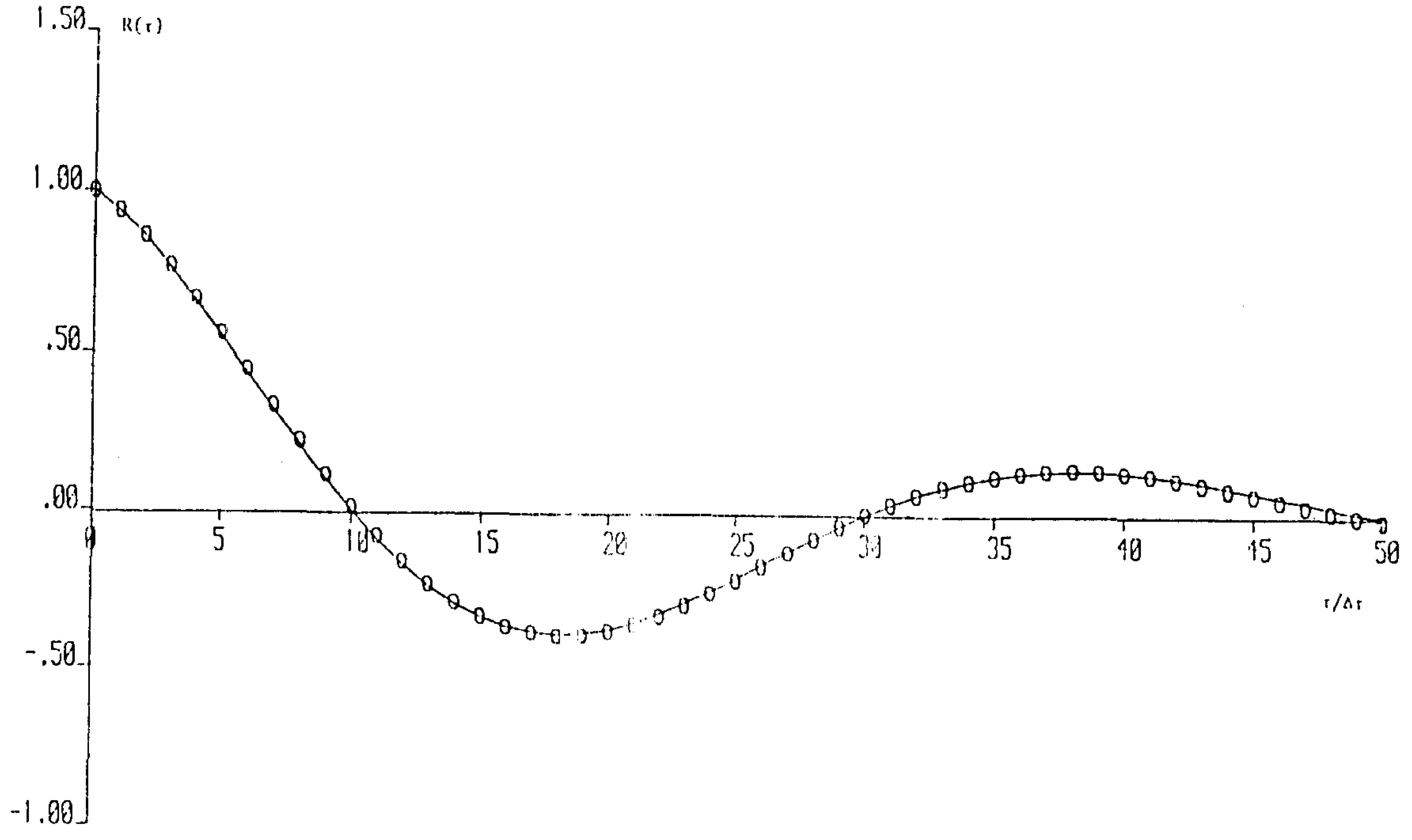

Figure 3- The random data with autocorrelation function:
$R(\tau) = \exp(-\tau) \cos(\pi\tau), \Delta\tau = 0.05$
sample size: N = 30000

Conclusion

This paper has considered the state space modelling and simulation of random signals with known, or desired, autocorrelation Functions. The method was based on passing white noise through linear filiters and designing the filter characteristics for a prescribed autocorrelation function. The state space approach was also employed to obtain a discrete model, suitable for simulations on digital computers. This was then examined by attempting to simulate a second order Gaussian process with a prescribed decaying cosine autocorrelation function, which showed to be successful.

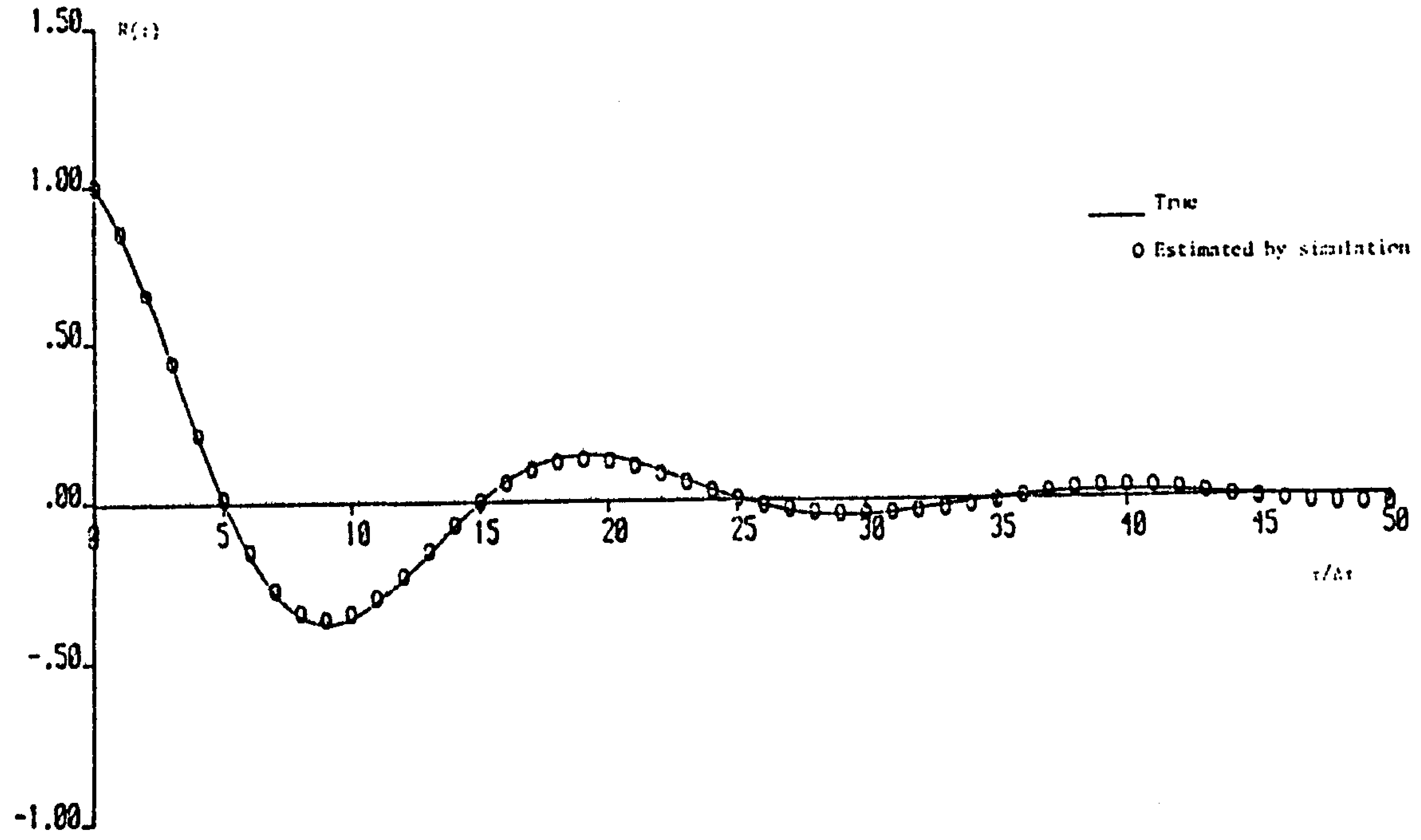

Figure 4- The random data with autocorrelation function:

$R(r) = \exp(-\tau)\cos(\pi\tau)$, $\Delta\tau = 0.1$

sample size: N = 30000

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